From Boltzmann–Gibbs ensemble to generalized ensembles

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Abstract

We reconsider the Boltzmann–Gibbs statistical ensemble in thermodynamics using the multinomial coefficient approach. We show that an ensemble is defined by the determination of four statistical quantities, the element probabilities p_i, the configuration probabilities P_j, the entropy S and the extremum constraints (EC). This distinction is of central importance for the understanding of the conditions under which a microcanonical, canonical and macrocanonical ensemble is defined. These three ensembles are characterized by the conservation of their sizes. A variation of the ensemble size creates difficulties in the definitions of the quadruplet $\{p_i, P_j, S, \text{EC}\}$, giving rise for a generalization of the Boltzmann-Gibbs formalism, such one as introduced by Tsallis. We demonstrate that generalized thermodynamics represent a transformation of ordinary thermodynamics in such a way that the energy of the system remains conserved. From our results it becomes evident that Tsallis's formalism is a very specific generalization, however, not the only one. We also revisit the Jaynes's Maximum Entropy Principle, showing that in general it can lead to incorrect results and consider the appropriate corrections.

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I. INTRODUCTION

For many years the Boltzmann-Gibbs (BG) theory of statistical thermodynamics was untouched. The assumption that matter consists of particles has led to the connection between statistics and thermodynamics through the construction of a *statistical ensemble*. An ensemble in physics is a collection of identical (described by the same variables and exposed to the same physical conditions) microscopic systems. We shall denote them in the current study as configurations. When the size of the ensemble approaches infinity (*thermodynamic limit*), we obtain the macroscopic picture of the matter, which is defined through a set of configuration probabilities P_j. Instead of considering each configuration probability separately, we can introduce a characteristic combination of all P_j for the entire ensemble. We call this combination *entropy* S of the system. Its analytical expression depends on the statistical features of the ensemble.

In the last two decades there has been a great effort to generalize the BG-theory, since it has been observed that there is a variety of systems [1] which can not be described sufficiently by the probability distributions issued from BG-statistics. First, Tsallis in 1988 introduced in his work [2] a possible generalization of the BG-entropy, the S_q^{T}-entropy, depending on one parameter q. For $q \to 1$ we obtain $S_q^{\mathrm{T}} \to S_{\mathrm{BG}}$. Since then, various studies on generalized entropy structures and their respective probability distributions have been done [3, 4, 5, 6, 7]. The configuration probability distributions in all cases were computed by applying the Jaynes's [8] *maximum entropy principle*. This principle sets the variation of a functional, with respect to the probabilities P_j, equal to zero. In Ref. [9] Tsallis showed thoroughly for the entropy S_q^{T} that the values of the parameter q describe correlations between the configuration probabilities. Although this study was only for the specific entropy S_q^{T}, the result about the role of the parameters can be extrapolated to every generalized entropy.

However, despite the mathematical developments in this field [10, 11, 12] (and many others) it is not yet clear what is the physical essence of this scientific direction. The aim of the current work is to unveil this physical essence of the generalized BG-theory, exploring the conditions of constructing a BG-ensemble via the *multinomial coefficient* approach and obtaining the respective configuration probabilities from a purely statistical point of view, without utilizing physical laws. After this is done, we correspond the mathematical results to

the analogous physical situation, presenting the borderline between ordinary and generalized thermostatistics. The Jaynes's maximum entropy principle is also under the loop. We show that this formalism, as it is applied up until now, is generally not correct, and we consider the appropriate corrections.

In Section II, we construct a random ensemble and derive the statistical quantities that characterize it. In Section III, we show how a modification in a random ensemble can change these statistical quantities. We present the condition under which the above statistical description breaks. In Section IV, we give the connection of our results to thermodynamics. We demonstrate the mathematical steps for the construction of a generalized BG-ensemble and its physical meaning. In Section V, we draw our main conclusions.

II. CONSTRUCTION OF A RANDOM ENSEMBLE

We consider a random system A of the size $N \to \infty$. Every space-position of the system is occupied by an element n_i. We define N_i as the frequency of the element n_i and W as the number of different types of elements. Then, the total number of the elements of the system is equal to its size $N = \sum_{i=1}^{W} N_i$. The probability of finding certain types of elements is given by $p_i = \lim_{N\to\infty} N_i/N$ under the normalization constraint $\sum_{i=1}^{W} p_i = 1$. We call them *element probabilities*. The statistical randomness in a system is defined through the independence of the elements n_i and consequently of their probabilities p_i as well.

Now, we want to organize the system A in a collection of configurations, each one of the size L. Then, the size of the system N can be written as

$$N = \lambda\, L, \tag{1}$$

where λ is the sum of all repetitions of each configuration. If E_j is the total number of elements of all repetitions in each configuration j, then N can be given also as $N = \sum_{j=1}^{\Omega(L)} E_j$, where $\Omega(L)$ is the maximum number of configurations without repetitions, depending on L. We denote $\Omega(L)$ as maximum configuration function. Furthermore, we demand that the collection procedure preserves the randomness feature for the configurations as well (independence). The independence between the configurations is projected on the existence of $\Omega(L)$. In general, when we have B elements consisting of W types, and B_i are the frequencies of each type of the elements, the number $\Omega(B)$ is obtained by computing the

multinomial coefficient $\mathrm{C}_{B_i}^{B}$ of the degree W:

$$\mathrm{C}_{B_i}^{B} := \frac{B!}{\prod_{i=1}^{W}(B_i)!} = B! \times \prod_{i=1}^{W}(1/B_i)!, \tag{2}$$

in the limit $B \to \infty$ for equal element probabilities. In our case the analogous quantity of B is L. However, if we consider the limit $L \to \infty$, it is easy to see that the feature of the statistical randomness of the ensemble does not hold, e.g. $L = N/2$ with $\lambda = 2$. What we actually wish to have is the inverse situation, $\lambda = N/2$ and $L = 2$. Yet, how can we compute then $\Omega(L)$? Since N and L are proportional, the idea is to compute the multinomial coefficient (2) with $B = N/\lambda$ and $B_i = N_i/\lambda$ in the limit $N \to \infty$. Accordingly, we have

$$\Omega_*(L,a) := \lim_{N\to\infty} \mathrm{C}_{p_i}^{N/\lambda} = \left[\prod_{i=1}^{W}(1/p_i)^{p_i}\right]^{L} = \mathrm{e}\left(L\sum_{i=1}^{W} p_i \ln(1/p_i)\right) = \mathrm{e}^{aL}, \tag{3}$$

where $0 \leqslant a := \sum_{i=1}^{W} p_i \ln(1/p_i) \leqslant \ln(W)$. Although, a has the structure of the Shannon-entropy [13], it is not an entropy but a constant, since the probabilities p_i for a given system are constant. $\Omega(L)$ is then defined when $\Omega_*(L,a)$ takes its maximum value ($a_{\max} = \ln(W)$):

$$\Omega(L) := \Omega_*(L, a_{\max}) = W^L. \tag{4}$$

In further, let us consider the probabilities P_j of each configuration j. If all configurations have the same probability P_j, then this is given by

$$P_j = \frac{1}{\Omega_*(N, a_{\max})} = \frac{1}{\Omega(L)} = W^{-L}, \tag{5}$$

with the normalization constraint $\sum_{j=1}^{\Omega(L)} P_j = 1$. Eq. (5) tells us that we can reach equal configuration probabilities P_j only if the element probabilities p_i are equal. If $p_i \neq p_k$, then the configurations are surely not equally probabilized, and the probabilities P_j present the structure

$$P_j := \left[\frac{1}{\Omega_*(N,a)}\right]^{f_j(p_i)} = \left[\prod_{i=1}^{W}(1/p_i)^{p_i}\right]^{-Lf_j(i)/p_i} = p_1^{Lf_j(1)} p_2^{Lf_j(2)} \cdots p_W^{Lf_j(W)}, \tag{6}$$

where $f_j(p_i) := f_j(i)/p_i$ is the characteristic number for each configuration. The constraints for $f_j(i)$ read

$$\sum_{i=1}^{W} f_j(i) = 1, \qquad \sum_{j=1}^{\Omega(L)} f_j(i) = W^{L-1}. \tag{7}$$

We call the sum over i's in Eq. (7) *repetition weight* of the configurations. Its connection with the repetitions of the possible configurations will become clear in equations to follow. In Table I we give a simple example for the values of $f_j(i)$ when $L = 2$ and $W = 2$ ($\Omega(L) = 4$). We are easily able to verify that the sum of the configuration probabilities P_j is equal to the unity, $\sum_{j=1}^{\Omega(L)} P_j = (p_1 + p_2)^L = 1$. For all values of W and L this equality takes the form

$j \rightarrow$	1	2	3	4
P_j	$p_1^{L\,f_1(1)} p_2^{L\,f_1(2)}$	$p_1^{L\,f_2(1)} p_2^{L\,f_2(2)}$	$p_1^{L\,f_3(1)} p_2^{L\,f_3(2)}$	$p_1^{L\,f_4(1)} p_2^{L\,f_4(2)}$
$f_j(1)$	1	$\frac{1}{2}$	$\frac{1}{2}$	0
$f_j(2)$	0	$\frac{1}{2}$	$\frac{1}{2}$	1

TABLE I: The configuration probabilities P_j and the terms $f_j(i)$ of the repetition weight for $W = 2$ and $L = 2$.

$$\sum_{j=1}^{\Omega(L)} P_j = \left[\sum_{i=1}^{W} p_i\right]^L . \tag{8}$$

Eq. (8) will be very useful later. Considering Eq. (7) we can prove the following relation

$$\sum_{j=1}^{\Omega(L)} P_j f_j(i) = p_i. \tag{9}$$

The probabilities in Eq. (6) can be also expressed in a different way. We rewrite P_j in exponential form and make the substitution $-\beta F_j = L \sum_{i=1}^{W} f_j(i) \ln(p_i)$. The negative sign is present because of the negative logarithmic term. β is a positive constant and F_j is now the new characteristic number of each configuration j. Then, P_j take the form

$$P_j = \frac{\mathrm{e}^{-\beta F_j}}{\sum_{j=1}^{\Omega(L)} \mathrm{e}^{-\beta F_j}}. \tag{10}$$

The structure of P_j in Eq. (10) is more familiar in thermodynamics.

Now, let us investigate the inverse situation. For the above given probability distribution P_j we construct the respective ensemble and search for an appropriate statistical quantity for its description. If M_j is the frequency of each type of configuration j and $M = \sum_{j=1}^{\Omega(L)} M_j$ is the total number of the configurations in the ensemble, then the probabilities P_j in Eq. (6) can be written as $P_j = \lim_{M\rightarrow\infty} M_j/M$. The number of all possible ensembles which consists

of the combination of M configurations, under the condition that P_j are independent, is given through the new multinomial coefficient of the degree $\Omega(L)$:

$$\mathrm{C}^M_{P_j} := \frac{M!}{\prod_{j=1}^{\Omega(L)}(M_j)!} = \frac{M!}{\prod_{j=1}^{\Omega(L)}(MP_j)!}. \tag{11}$$

Considering Eq. (11) in the limit $M \to \infty$ we obtain the analogy to Eq. (3)

$$\lim_{M\to\infty} \mathrm{C}^M_{P_j} = \left[\prod_{j=1}^{\Omega(L)} (1/P_j)^{P_j}\right]^M = \mathrm{e}\left(M \sum_{j=1}^{\Omega(L)} P_j \ln(1/P_j)\right). \tag{12}$$

Then, we define the following statistical quantity

$$S_{\mathrm{Sh}}(L) := \lim_{M\to\infty} \frac{\ln\left(\mathrm{C}^M_{P_j}\right)}{M} = \sum_{j=1}^{\Omega(L)} P_j \ln(1/P_j), \tag{13}$$

which is the well known Shannon-entropy of the ensemble. The Shannon entropy presents two very interesting and useful properties for a random system. First, if we replace P_j of Eq. (6) in the above equation, using Eq. (9), we obtain

$$S_{\mathrm{Sh}}(L) = a\,L = L\sum_{i=1}^{W} p_i \ln(1/p_i). \tag{14}$$

As we can see, exponential distribution functions make Shannon entropy linear with respect to L, and because of Eq. (1), with respect to N. We denote the exponential distribution functions which make an entropy extensive as $\overline{P}_j$. Second, computing the derivation of the entropy (13) with respect to P_j, replacing P_j through $\overline{P}_j$ of Eq. (10) and then integrating with respect to P_j we obtain

$$\int_0^{P_j} \left.\frac{\partial S_{\mathrm{Sh}}(P_j)}{\partial P_j}\right|_{P_j=\overline{P}_j} dP'_j = \alpha \underbrace{\sum_{j=1}^{\Omega(L)} P_j}_{\mathrm{NC}} + \beta \underbrace{\sum_{j=1}^{\Omega(L)} F_j P_j}_{\mathrm{MVC}}, \tag{15}$$

where $\alpha := \ln\left(\sum_{k=1}^{\Omega(L)} \mathrm{e}^{-\beta F_k}\right) - 1$ is a constant with respect to the index j. We call the terms in Eq. (15) *Extremum Constraints* (EC). These are composed of the normalization constraint (NC) and the mean value constraint (MVC). At this point we have introduced all appropriate statistical quantities for the description of a random ensemble. These quantities are given in the quadruplet $\{p_i,\, P_j,\, S_{\mathrm{Sh}},\, \mathrm{EC}\}$ or equivalent $\{\Omega,\, P_j,\, S_{\mathrm{Sh}},\, \mathrm{EC}\}$. We would like to emphasize here that all four quantities depend structurally on each other.

III. MODIFICATIONS IN A RANDOM ENSEMBLE

We want to explore how the quadruplet $\{\Omega,\, P_j,\, S_{\mathrm{Sh}},\, \mathrm{EC}\}$ changes if we insert random modifications in a random ensemble (the random modifications assure the conservation of the first term in the above quadruplet, Ω, since we do not want to change the dynamic but to consider some perturbations of a purely random state). Therefore, let us consider the random test-ensemble A for $L = 2$, $W = 2$ and $N = 24$, as demonstrated in Table II. The element probabilities p_0 and p_1 are equal $p := p_0 = p_1 = 1/2$ and the configuration

Ensemble A		Ensemble B
$\begin{pmatrix} (00) & (00) & (00) \\ (01) & (\mathbf{0}1) & (01) \\ (10) & (\mathbf{1}0) & (10) \\ (11) & (11) & (11) \end{pmatrix}$	$\xrightarrow[\mathbf{1}\uparrow]{\mathbf{0}\downarrow}$	$\begin{pmatrix} (00) & (00) & (00) \\ (01) & (\mathbf{1}1) & (01) \\ (10) & (\mathbf{0}0) & (10) \\ (11) & (11) & (11) \end{pmatrix}$
$p_0 = p_1 = \frac{1}{2}$	$\longrightarrow$	$p_0 = p_1 = \frac{1}{2}$
$P_{00} = \frac{1}{4} \times (1)$, $P_{01} = \frac{1}{4} \times (1)$	$\longrightarrow$	$P_{00} = \frac{1}{4} \times \left(\frac{4}{3}\right)$, $P_{01} = \frac{1}{4} \times \left(\frac{4}{6}\right)$
$P_{10} = \frac{1}{4} \times (1)$, $P_{11} = \frac{1}{4} \times (1)$	$\longrightarrow$	$P_{10} = \frac{1}{4} \times \left(\frac{4}{6}\right)$, $P_{11} = \frac{1}{4} \times \left(\frac{4}{3}\right)$

TABLE II: Changes in the random ensemble A and accordingly in its configuration probabilities.

probabilities $\{P_j\}_{j=00,\,01,\,10,\,11}$ as well, $P_{00} = P_{01} = P_{10} = P_{11} = 1/4 = p^L$. We now change the position of two elements, as shown in the table, creating in this way the new ensemble B. The element probabilities p_i remain the same in the new ensemble but the configuration probabilities are different. Thus, the ensemble B is not a pure random ensemble but a pseudo-random ensemble. In order to capture the changes in the probabilities P_j we extend the first relation in Eq. (7) as follows

$$\sum_{i=1}^{W} f_j(i) = 1 + \frac{\ln(1/c_j)}{\ln(\Omega(L))}, \tag{16}$$

where c_j are a positive constants. Then, for equal probabilities p_i, we obtain

$$P_j = p^L\, c_j. \tag{17}$$

The normalization constraint of P_j's leads us to the definition of c_j's through the relation $\sum_{j=1}^{\Omega(L)} c_j = \Omega(L)$. As we can verify from Eq. (17) and the above table, for a random ensemble

c_j are equal to unity. For a pseudo-random ensemble (or modified random ensemble) we have $c_j \neq 1$. At this point, it becomes clear why we called the sum in Eq. (16) repetition weight. Eq. (17) (or Eq. (16)) reproduces the probability results presented in Table II. When we organize a system in an ensemble we can succeed in having the repetition weight equal to unity. Yet, when we modify an already organized system, then it is most probable that the repetition weight is different from the unity. Because of the introduction of the terms c_j, Eq. (9) takes the form

$$\sum_{j=1}^{\Omega(L)} P_j f_j(i) = p_i \left(1 + \frac{\sum_{j=1}^{\Omega(L)} P_j \ln(1/c_j)}{\ln(\Omega(L))}\right). \tag{18}$$

Accordingly, the linear relation 14 between $S_{\text{Sh}}(L)$ and L tends to

$$S_{\text{Sh}}(L) = a\,L + e_L, \tag{19}$$

where $e_L := \frac{a}{\ln(W)} \sum_{j=1}^{\Omega(L)} P_j \ln(1/c_j)$. Usually, e_L is very small and thus negligible. However, even if it is not small enough, it is still negligible for $L \gg 1$, since the probabilities P_j decay exponentially with respect to L. Finally, it can be completely eliminated by the transformation $f_j(i) - p_i \frac{\ln(1/c_j)}{\ln(\Omega(L))} \to \tilde{f}_j(i)$, where $\sum_{i=1}^{W} \tilde{f}_j(i) = 1$ is the new repetition weight. Thus, the Shannon entropy can remain linear with respect to L even when $c_j \neq 1$ and consequently, because of Eq. (1), with respect to N.

Since, the probabilities P_j change by any modification in the configurations, it is clear that F_j's in Eq. (10) change as well. In Table III, we show under what conditions F_j changes and what structure it takes. Let us consider, at first, a random ensemble ($c_j = 1$) with equal probabilities p_i, P_j. Then, the values F_j are constant with respect to the index j and equal to $F_j = \ln(\Omega(L))/\beta$. We denote this situation as *ensemble of Type I.* Our next step is to exchange elements n_i randomly between the configuration repetitions. As previously mentioned, in this case we have generally $c_j \neq 1$. Then, we can observe changes in E_j, thus $F_j = E_j$. We denote this situation as *ensemble of Type II.* If the element exchange takes place between the ensemble and a reservoir of elements, in general N_i changes. Then, the probabilities P_j must be extended to an N_i-dependence and accordingly the structure of F_j tends to $F_{i,j} = E_j + \mu N_i$, where μ is a proportionality constant. The normalization factor in Eq. (10) changes as well to $1/\sum_{i,j} \mathrm{e}^{-\beta F_{i,j}}$. We denote this situation as *ensemble of Type III.* In these three described cases the total number of elements N, or in other words the size of the system, remains conserved. In the last case, we consider again the latter ensemble,

Ensemble Type→	I	II	III	IV
N	$\lambda\, L$	$\lambda\, L$	$\lambda\, L$	$\lambda(N, L)$
constants	N, N_i, c_j	N, N_i	N	–
variables	–	c_j	N_i, c_j	N, N_i, c_j
F_j	$L \ln(W^{1/\beta})$	E_j	$E_j + \mu N_i$	$\boxed{?}$

TABLE III: Statistical features of a modified random ensemble depending on the nature of the modifications.

but now we allow also a variation of N of the ensemble. Then, N and L are not connected through Eq. (1) but through a relation of the form $N = \lambda(N, L)$. If the last expression can not be solved analytically with respect to L, then the relations we have derived for $\Omega(L)$, P_j, S_{Sh} or EC are no longer valid. Consequently, we have observed that random modifications do not cause changes in the quadruplet $\{\Omega,\, P_j,\, S_{\text{Sh}},\, \text{EC}\}$ as long as the size of the ensemble N is conserved.

Which one of the four above mentioned situations represents a *statistical equilibrium*? Mathematically, the answer is not strictly defined. It depends on the conservation law we prefer to emphasize, c_j, N_i or N. We know only that when we have ensembles of Type IV all quantities vary, and thus the last situation can not be considered as a defining possibility of a statistical equilibrium.

IV. CONNECTION TO THERMODYNAMICS

By assuming that the elements n_i correspond to physical particles, the quantity E_j to the energy of each configuration, the constant μ to the chemical potential of the ensemble, the expression $\mathrm{e}(-\beta E_j)$ to the Boltzmann-factor and the modifications to physical interactions, we obtain successfully the connection between the above approach and ordinary statistical thermodynamics. The entropy (13) is then called Boltzmann-Gibbs entropy. In thermodynamics the equilibrium state of a system under consideration is defined by the conservation of N. This definition includes the ensembles of Type I - III which are denoted in literature as *microcanonical*, *canonical* and *macrocanonical* BG-ensemble, respectively. Eq. (14) tells us that in a BG-equilibrium state the BG-entropy of the system does not take necessarily

its maximum value ($p_i = p_k$ and $c_j = 1$) but it becomes *extensive* (linear) with respect to the size of the system.

As we have seen, in the case of an ensemble of Type IV N is not conserved Ensembles of this type correspond to open interacting thermodynamical systems. Is the statistical description of such an ensemble possible? Generally, it is not easy to answer this question. Usually, when we explore a system we do not know from the start its statistical features which are projected on the probability distribution sets $\{p_i\}_{i=1,2\ldots,W}$ and $\{P_j\}_{j=1,2,\ldots,\Omega(L)}$. Thus, there is a great difficulty to construct the appropriate multinomial coefficient from which we can obtain the quadruplet of the system $\{\Omega,\, P_j,\, S,\, \mathrm{EC}\}$.

A possible way to overcome this difficulty is to generalize the multinomial coefficient in Eq. (2), or in other words, to consider correlated probabilities p_i and P_j, separately. The multinomial coefficient (2) is based on two operations, multiplication and division $\{\times, /\}$. So, the idea is to generalize these two operations in order to obtain a generalized multinomial coefficient. A generalized multiplication or division $\{\otimes_{\mathcal{Q}},\, \oslash_{\mathcal{Q}}\}$ depends on a parameter set $\mathcal{Q} = \{Q_i\}_{i=1\ldots m}$ and tends to the ordinary ones for certain values $\mathcal{Q}_0$ of the parameters, $\{\otimes_{\mathcal{Q}},\, \oslash_{\mathcal{Q}}\} \to \{\otimes_{\mathcal{Q}_0},\, \oslash_{\mathcal{Q}_0}\} = \{\times,\, /\}$. How can we generalize multiplication and division? For the elements $x,\, y \in \mathbb{R}_+$ these operations can be written as

$$x \otimes_{\mathcal{Q}_0} y = x \times y = \mathrm{e}\left(\ln(x) + \ln(y)\right), \tag{20}$$

$$x \oslash_{\mathcal{Q}_0} y = \; x \,/\, y \; = \mathrm{e}\left(\ln(x) - \ln(y)\right). \tag{21}$$

Here, it becomes obvious that the generalization of the operations $\{\times, /\}$ can be succeeded through deformed logarithmic and exponential functions. Thus, we have

$$x \otimes_{\mathcal{Q}} y = \mathrm{e}_{\mathcal{Q}}\left(\ln_{\mathcal{Q}}(x) + \ln_{\mathcal{Q}}(y)\right), \tag{22}$$

$$x \oslash_{\mathcal{Q}} y = \mathrm{e}_{\mathcal{Q}}\left(\ln_{\mathcal{Q}}(x) - \ln_{\mathcal{Q}}(y)\right). \tag{23}$$

with $\mathrm{e}_{\mathcal{Q}_0}(x) = \mathrm{e}^x$ and $\ln_{\mathcal{Q}_0}(x) = \ln(x)$. In the definition of $\ln_{\mathcal{Q}}(x)$ (or $\mathrm{e}_{\mathcal{Q}}(x)$) we demand that the equality $\ln_{\mathcal{Q}}(1) = 0$ (or $\mathrm{e}_{\mathcal{Q}}(0) = 1$) is fulfilled. In this way we obtain new maximum configuration functions $\Omega_{\mathcal{Q}}(L)$, new configuration probability distributions $P_j^{(\mathcal{Q})}$, new entropy structures $S_{\mathcal{Q}}$ and new extremum constraints $\mathrm{EC}_{\mathcal{Q}}$. All these new quantities constitute a *generalized Boltzmann-Gibbs ensemble*. So, within this generalization we are able to find a $\mathcal{Q}$-probability distribution which makes the respective $\mathcal{Q}$-entropy extensive with respect to the size of the system N (or L). What is the physical essence of the generalization

procedure described above? We consider an ensemble, whose total number of particles, for some entropies is not conserved and for some others is conserved. It becomes evident, that a generalized BG-ensemble is a transformed BG-ensemble. The values of the parameters $\mathcal{Q}$ are characteristic for every transformed ensemble. Indeed, if we transform the probabilities P_j in BG-entropy as follows

$$P_j \to -\frac{P_j^{(\mathcal{Q})} \ln_{\mathcal{Q}}(1/P_j^{(\mathcal{Q})})}{W\left[P_j^{(\mathcal{Q})} \ln_{\mathcal{Q}}(1/P_j^{(\mathcal{Q})})\right]}, \tag{24}$$

where $W[x]$ is the Lambert function [19], and the probabilities p_i in such a way that $\Omega(L) \to \Omega_{\mathcal{Q}}(L)$, then we obtain any generalized trace-form entropy $S_{\mathcal{Q}} = \sum_{j=1}^{\Omega_{\mathcal{Q}}(L)} P_j^{(\mathcal{Q})} \ln_{\mathcal{Q}}(1/P_j^{(\mathcal{Q})})$. For non-trace-form entropies we transform a function of the probabilities P_j, $H(P_j)$. In the transformed ensemble the normalization condition for the new probabilities $P_j^{(\mathcal{Q})}$ is given by the generalization of Eq. (8) as follows:

$$\sum_{j=1}^{\Omega_{\mathcal{Q}}(L)} P_j^{(\mathcal{Q})} = \left(\sum_{i=1}^{W} p_i\right)^{\otimes_{\mathcal{Q}}^{L}} = 1, \tag{25}$$

where the generalized exponent, according to Eq. (22), has the form $x^{\otimes_{\mathcal{Q}}^{a}} = \mathrm{e}_{\mathcal{Q}}\left(a \ln_{\mathcal{Q}}(x)\right)$. The probabilities P_j are not normalized in the transformed ensemble, since $\Omega(L)$ is also transformed. The question that arises from the ensemble-transformation is whether the physics of the BG-ensemble, which is projected on the definition of the extremum constraints (15), does not change. We shall return to this point in the next paragraph.

Up to now, all proposals for a possible generalization of the BG-statistics are mostly based only on the generalization of the BG-entropy structure. By applying the Jaynes's formalism, one could derive the respective probability distributions using *ad hoc* [3, 7, 14, 15, 16] the extremum constraints in Eq. (15):

$$\frac{\partial S_{\mathcal{Q}}(P_j^{(\mathcal{Q})})}{\partial P_j^{(\mathcal{Q})}} = \sum_{j=1}^{\Omega(L)} \left(\alpha + \beta\, F_j\right). \tag{26}$$

where α and β correspond to the Lagrangian multiplicators. However, we showed in Eq. (15) that the structure of the right hand side of Eq. (26) is valid only for the probabilities (6) and the entropy (13). So, the use of the formalism in Eq. (26) for an arbitrary entropy structure leads in general to incorrect results. Comparing Eqs. (15) and (26) we see that the correct expression of the variation functional I in Jaynes's maximum entropy principle

is of the form

$$I := S_{\mathcal{Q}}(P_j^{(\mathcal{Q})}) - \int_0^{P_j^{(\mathcal{Q})}} \frac{\partial S_{\mathcal{Q}}(P_j^{\prime(\mathcal{Q})})}{\partial P_j^{\prime(\mathcal{Q})}} \Bigg|_{P_j^{\prime(\mathcal{Q})}=\overline{P}_j^{\prime(\mathcal{Q})}} dP_j^{\prime(\mathcal{Q})}. \tag{27}$$

In other words, in general we can not compute the maximum entropy probabilities (configuration probabilities) $P_j^{(\mathcal{Q})}$ from Jaynes's principle because we first need to know them in order to obtain the extremum constraints. Very recently, in Ref. [17] the present author has constructed two generalized multinomial coefficients and shown, among others, that the probability distribution function for which the Rényi entropy [18] becomes extensive is an ordinary exponential function and not a q-exponential one [6], which is derived from Eq. (26). On the other hand, it is remarkable that in the case of Tsallis entropy the Jaynes's formalism (26) does give the correct results [17]. Let us shed light on this point. The inner structure of the BG-entropy has the expression $x\ln(1/x)$. From the derivation of this expression we obtain

$$\frac{\partial}{\partial x}\left[x\ln(1/x)\right] = \ln(1/x) - 1. \tag{28}$$

Then, the Jaynes's formalism in Eq. (26) can lead to correct results only if the entropy is of trace-form, $S_{\mathcal{Q}} = \sum_{j=1}^{\Omega_{\mathcal{Q}}(L)} P_j^{(\mathcal{Q})} \ln_{\mathcal{Q}}(1/P_j^{(\mathcal{Q})})$, and the following relation is fulfilled

$$\frac{\partial}{\partial x}\left[x\ln_{\mathcal{Q}}(1/x)\right] = \lambda_1(\mathcal{Q})\ln_{\mathcal{Q}}(1/x) + \lambda_2(\mathcal{Q}), \tag{29}$$

where $\lambda_1(\mathcal{Q})$, $\lambda_2(\mathcal{Q})$ are constants with $\lambda_1(\mathcal{Q}_0) = -\lambda_2(\mathcal{Q}_0) = 1$. Solving the differential equation (29) and considering the appropriate boundary conditions, we obtain the generalized logarithm

$$\ln_{\mathcal{Q}}(x) = \frac{x^{1-\lambda_1(\mathcal{Q})} - 1}{1-\lambda_1(\mathcal{Q})}. \tag{30}$$

Accordingly, this is the only logarithmic-like function for the entropy $S_{\mathcal{Q}}$ which preserves the structure of the right hand side in Eq. (26) (or the EC in Eq. (15)). When the transformation of the BG-ensemble is based on the deformed logarithm (30) and the obtained entropy is of trace-form, then the physics of the BG-ensemble does not change after the transformation. Indeed, one can verify, that the transformation rule in Eq. (24) is only invertible when the deformed logarithm has the form (30). Considering only one parameter $\mathcal{Q} = \lambda_1(\mathcal{Q}) = q$ in Eq. (30), the obtained entropy based on the deformed logarithm $\ln_q(x)$ is the Tsallis entropy.

V. CONCLUSIONS

We have shown that a statistical ensemble is defined through four quantities, the element probabilities p_i, which refer to the ensemble elements n_i and lead to the maximum number of all possible configurations $\Omega(L)$, the configuration probabilities P_j, the entropy S and the extremum constraints (EC) of the ensemble, like the probability normalization and the mean value constraints. The explicit structure of these quantities for a random ensemble is derived. The structure of each quantity of the quadruple $\{\Omega,\, P_j,\, S_{\rm Sh},\, {\rm EC}\}$ depends on the structure of the other quantities. We demonstrated that non of these structures changes under random modifications in the ensemble if the size of the ensemble N is conserved. It is well known that a random ensemble corresponds to the Boltzmann-Gibbs ensemble in physics. So, we corresponded our results to statistical thermodynamics by assuming that the elements n_i represent physical particles. When the total number of particles N in the Boltzmann-Gibbs ensemble is conserved, we have statistical Boltzmann-Gibbs equilibrium and the Boltzmann-Gibbs entropy becomes extensive with respect to N. If N is not conserved then the Boltzmann-Gibbs entropy can not be defined and the appropriate entropy is to be found in the area of generalized Boltzmann-Gibbs statistics, introduced first by Tsallis. We demonstrated that generalized ensembles based on deformed logarithmic and exponential function transform the Boltzmann-Gibbs ensemble in such a way that N is conserved. This result is of great importance in physics because the conservation of the size of the Boltzmann-Gibbs ensemble is equivalent to the conservation of its energy E_j, since $N = \sum_{j=1}^{\Omega(L)} E_j$. From our approach becomes clear that the parameters of the generalized ensembles, which represent correlations between the probabilities, do not describe correlations that we detect but correlations that we create, so that we obtain $N = conserved$. Generalized ensembles based on the deformed logarithm (30) do not change the physics of the Boltzmann-Gibbs ensemble after the transformation.

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